MATHEMATICAL DESCRIPTION OF THE BEHAVIOR OF AN ELASTIC ISOTROPIC SOLID BY MEANS OF A PIECEWISE LINEAR POTENTIAL

(O MATEMATICHESKOM OPISANII POVEDENIIA UPRUGOGO IZOTROPNOGO TELA PRI POMOSHCHI KUSOCHNO-LINEINOGO POTENTSIALA)

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The relation between stress and strain components for an elastic isotropic body in the region of small strains is considered when there exists a potential which is piecewise linear.

It is assumed that a linear Hooke's Law holds for uniaxial tension-compression and for pure shear, and that volume changes are proportional to the mean stress. The behavior of the model differs in general from the behavior of an elastic isotropic body obeying the linear theory of elasticity [1,2].

1. For determination of the relations between stress and strain for an elastic body we start with the expressions

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}} \tag{1.1}$$

Here σ_{ij} and ϵ_{ij} are the stress and strain components respectively, and U is the potential of the strains.

For isotropic bodies the potential U is a function of the invariants of the stress tensor

$$U = U(\sigma, \Sigma_2, \Sigma_3) \tag{1.2}$$

where σ is the first invariant of the stress tensor, Σ_2 and Σ_3 are the respective second and third invariants of the stress deviator tensor.

We assume that a linear relation exists between the mean stress σ and the volume strain ϵ

$$\sigma = 3K\varepsilon \quad \begin{matrix} \varepsilon = {}^1/_3\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma = {}^1/_3\,(\sigma_x + \sigma_y + \sigma_z) \end{matrix} \quad (K = \mathrm{const}) \tag{1.3}$$

(Here K is the volume modulus of compressibility.)

In order for (1.3) to hold, one must set

$$U = \frac{\sigma^2}{2K} + \Phi\,(\Sigma_2, \Sigma_3) \tag{1.4}$$

The relation between stresses and strains determined from (1.4) in accordance with (1.1) will satisfy relations (1.3).

We assume that with a change in sign of the stress the reversed strain components likewise suffer only a change of sign. It follows that the function Φ depends on the modulus $|\Sigma_3|$.

In the field of principal stresses σ_1, σ_2, σ_3 the function Φ is interpreted as the totality of cylindrical surfaces with equal values of Φ, the generators of which are parallel to the axes $\sigma_1 = \sigma_2 = \sigma_3$.

We consider a deviator plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$, on which $U = \Phi$. The curves of intersection of the U-surfaces and the deviator plane we denote by the term "potential curves".

If the value of the potential is determined for uniaxial tension-compression, then the possible nonconcave potential curves will lie between the hexagons $ABCDEF$ and $A_1B_1C_1D_1E_1F_1$ (Fig. 1a).

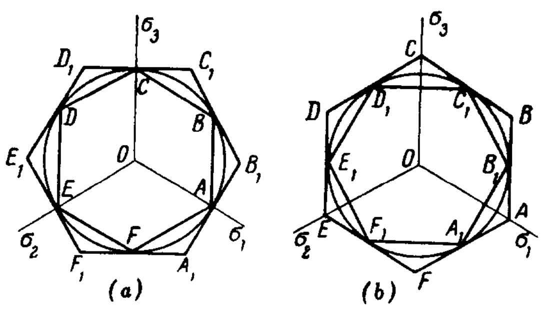

Fig. 1.

If the value of the potential is determined from an experiment in pure shear, the mutual disposition of the hexagons is shown in Fig. 1b.

It is well known that the function Φ is not completely determined from the results of simple tests (the connection between stresses and strains for tension-compression, pure shear, etc.) and so, generally speaking, one may construct as many relations as desired between the components of stress and strain for an elastic isotropic body leading to a linear Hooke's Law for uniaxial tension-compression [3,4].

The linear relations of the generalized Hooke's Law are obtained from the special assumption that $\Phi = \Phi(\Sigma_2)$. The corresponding potential curve on Fig. 1 is represented by the circle.

We now consider the relations given by the theory for an elastic isotropic body when the potential curves are represented by hexagons similar to the hexagon $ABCDEF$ in Fig. 1.

The surface of the function Φ in the principal stress field is piecewise smooth; hence, a generalization of the determination of (1.1) is necessary. It follows from the relations (1.1) by the use of (1.3) and (1.4) that the components of the tensor of strain deviators will be orthogonal to the surface Φ in the stress-strain field. In other words, the components $\epsilon_{ij} - \delta_{ij}\epsilon$ ($\delta_{ij} = 1$; $\delta_{ij} = 0$, $i \neq j$) coincide with the normal to a surface which is tangent to Φ at a given point.

The surface Φ may be interpreted as a bending of both tangents to the plane. Particular points and lines of the surface Φ are interpreted as the limits of smooth sequences.

For continuous passage from a smooth potential surface to one which is piecewise smooth, we find that the potential surface of the strain tensor may take different values at certain points. If this singularity is represented by the tangents to the smooth surfaces

$$\Phi_1 = \Phi_2 = \ldots = \Phi_m = \Phi \tag{1.5}$$

then the set of components of the strain tensor may be presented in the form

$$\varepsilon_{ij} - \delta_{ij}\varepsilon = \sum_{k=1}^{m} \lambda_k \frac{\partial \Phi_k}{\partial \sigma_{ij}} \tag{1.6}$$

Since $\lambda_1 + \lambda_2 + \ldots \lambda_m = 1$, $\lambda_k \geqslant 0$, one obtains $m - 1$ possible strain components in the parametric family. Just as in the theory of plasticity [5-8] one has to rely on an additional condition applicable to the specific problem (edge, initial, or other condition) in order to determine the state of strain completely.

Upon multiplying relation (1.6) by $d\sigma_{ij}$, we obtain

$$\sum_{ij} \varepsilon_{ij} d\sigma_{ij} = \varepsilon d\sigma + \sum_{k=1}^{m} \lambda_k d\Phi_k \tag{1.7}$$

It follows from (1.5) that $d\,\Phi_1 = d\,\Phi_2 = \ldots\, d\,\Phi_n = d\,\Phi$, and so we find from (1.7)

$$\sum_{ij} \varepsilon_{ij} d\sigma_{ij} = dU$$

Relations analogous to (1.6) are obtained from the theory of the generalized plastic potential [5,6].

2. We consider the relations for the theory of an elastic isotropic body (Fig. 1) with a piecewise linear potential $ABCDEF$. Evidently maximum freedom in the strain occurs at the peaks. Any arbitrarily directed strain may correspond to the peaks of the hexagon $ABCDEF$; or, in other words, an arbitrary state of strain may correspond to a state of stress with maxima on the given hexagon peaks.

Considering the side AB (Fig. 1), we shall have for it

$$U = \frac{\sigma^2}{2K} + \Phi(\xi), \qquad \xi = \sigma_1 - \sigma_2 \tag{2.1}$$

In accordance with (1.1) we obtain from (2.1)

$$\varepsilon_1 - \varepsilon = \frac{\partial \Phi}{\partial \xi}, \qquad \varepsilon_2 - \varepsilon = -\frac{\partial \Phi}{\partial \xi}, \quad \varepsilon_3 = -\varepsilon = 0 \tag{2.2}$$

In the case of uniaxial tension-compression Hooke's Law holds by assumption:

$$\sigma_1 = E\varepsilon_1, \qquad \sigma_2 = \sigma_3 = 0 \tag{2.3}$$

For (2.3) to be true, it follows from (2.2) and (1.3) that one must set

$$\Phi = \frac{1}{2}\left(\frac{1}{E} - \frac{1}{9K}\right)(\sigma_1 - \sigma_2)^2 \tag{2.4}$$

By denoting $1/G = (1/E - 1/9K)$, we write Expression (2.4) in the form

$$\Phi = \frac{2}{3G}\tau_{\max}^2, \qquad \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} \tag{2.5}$$

Expressions for the function Φ for the other sides of the hexagon $ABCDEF$ (Fig. 1) may be written by analogy.

Consider the relations corresponding to the peaks of the hexagon (Fig. 1). The peak A is the point of intersection of the sides AB and AF. We have for these sides

$$U_{AB} = \frac{\sigma^2}{2K} + \frac{1}{6G}(\sigma_1 - \sigma_2)^2, \qquad U_{AF} = \frac{\sigma^2}{2K} + \frac{1}{6G}(\sigma_1 - \sigma_3)^2 \tag{2.6}$$

We obtain for the peak A in accordance with (1.6) and (2.6)

$$\begin{aligned} \varepsilon_1 - \varepsilon &= \frac{1}{3G}[\lambda_1(\sigma_1 - \sigma_2) + \lambda_2(\sigma_1 - \sigma_3)] \\ \varepsilon_2 - \varepsilon &= -\frac{\lambda_1}{3G}(\sigma_1 - \sigma_2) \qquad (\lambda_1 + \lambda_2 = 1) \\ \varepsilon_3 - \varepsilon &= -\frac{\lambda_2}{3G}(\sigma_1 - \sigma_3) \qquad (\lambda_1 \geqslant 0,\ \lambda_2 \geqslant 0) \end{aligned} \tag{2.7}$$

By taking into account that $\sigma_2 = \sigma_3$ at the peak A, we rewrite (2.7) in the form

$$\begin{gathered} \varepsilon_1 - \varepsilon = \frac{T}{3G}, \qquad \varepsilon_2 - \varepsilon = -\frac{\lambda_1}{3G}T \\ \varepsilon_3 - \varepsilon = -\frac{\lambda_2}{3G}T \\ (T = \sigma_1 - \sigma_2 = \sigma_1 - \sigma_3) \end{gathered} \tag{2.8}$$

It is apparent from (2.8) that Hooke's Law (2.3) holds. We write down the initial conditions for the deviator components in the xyz-coordinate system. We suppose that the xyz-axes make angles with the directions of the principal stresses designated by 1, 2, 3, the cosines of which are here tabulated.

TABLE

	1	2	3
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

From the relations

$$\begin{aligned} \sigma_x &= \sigma_1 l_1^2 + \sigma_2 m_1^2 + \sigma_3 n_1^2, \ldots \\ \tau_{xy} &= \sigma_1 l_1 l_2 + \sigma_2 m_1 m_2 + \sigma_3 n_1 n_2, \ldots \end{aligned}$$

we obtain at the peak A

$$\begin{aligned} \sigma_x &= \sigma - \tfrac{1}{3}T + T\cos^2\theta_1, \qquad & \tau_{xy} &= T\cos\theta_1\cos\theta_2 \\ \sigma_y &= \sigma - \tfrac{1}{3}T + T\cos^2\theta_2, \qquad & \tau_{yz} &= T\cos\theta_2\cos\theta_3 \\ \sigma_z &= \sigma - \tfrac{1}{3}T + T\cos^2\theta_3, \qquad & \tau_{zx} &= T\cos\theta_3\cos\theta_1 \end{aligned} \tag{2.9}$$

Here θ_1, θ_2, θ_3 are the angles determining the direction of the principal stress σ_1 in the xyz-coordinate system.

In particular, it follows from (2.9) that

$$
\begin{aligned}
&\left(\sigma_x-\sigma+\tfrac{1}{3}T\right)\left(\sigma_y-\sigma+\tfrac{1}{3}T\right)-\tau_{xy}^2=0\\
&\left(\sigma_y-\sigma+\tfrac{1}{3}T\right)\left(\sigma_z-\sigma+\tfrac{1}{3}T\right)-\tau_{yz}^2=0\\
&\left(\sigma_z-\sigma+\tfrac{1}{3}T\right)\left(\sigma_x-\sigma+\tfrac{1}{3}T\right)-\tau_{zx}^2=0
\end{aligned}
\tag{2.10}
$$

From (2.10) we find that

$$
\begin{aligned}
T&=\tfrac{3}{2}\left[s_x+\sqrt{s_x^2-4\left(s_y s_z-\tau_{yz}^2\right)}\right]\\
T&=\tfrac{3}{2}\left[s_y+\sqrt{s_y^2-4\left(s_z s_x-\tau_{zx}^2\right)}\right]\\
T&=\tfrac{3}{2}\left[s_z+\sqrt{s_z^2-4\left(s_x s_y-\tau_{xy}^2\right)}\right]\\
&\qquad (s_{ij}=\sigma_{ij}-\delta_{ij}\sigma)
\end{aligned}
\tag{2.11}
$$

Each radical in (2.11) requires a plus sign since $T > 0$.

From the condition of isotropy, by asserting the coincidence of the principal directions of stress and strain following [7], we obtain

$$
\begin{aligned}
\varepsilon_x+\varepsilon_{xy}\frac{\cos\theta_2}{\cos\theta_1}+\varepsilon_{xz}\frac{\cos\theta_3}{\cos\theta_1}&=\varepsilon_{xy}\frac{\cos\theta_1}{\cos\theta_2}+\varepsilon_y+\varepsilon_{yz}\frac{\cos\theta_3}{\cos\theta_2}\\
&=\varepsilon_{xz}\frac{\cos\theta_1}{\cos\theta_2}+\varepsilon_{yz}\frac{\cos\theta_2}{\cos\theta_3}+\varepsilon_z=\varepsilon_1
\end{aligned}
\tag{2.12}
$$

From this we find after making use of (2.9), (2.10) and (2.11)

$$
\begin{aligned}
&\varepsilon_x+\varepsilon_{xy}\frac{\sigma_y-\sigma+{}^1/_3T}{\tau_{xy}}+\varepsilon_{xz}\frac{\sigma_z-\sigma+{}^1/_3T}{\tau_{xz}}=\varepsilon_{xy}\frac{\sigma_x-\sigma+{}^1/_3T}{\tau_{xy}}+\\
&+\varepsilon_y+\varepsilon_{yz}\frac{\sigma_z-\sigma+{}^1/_3T}{\tau_{yz}}=\varepsilon_{xz}\frac{\sigma_x-\sigma+{}^1/_3T}{\tau_{xz}}+\varepsilon_{yz}\frac{\sigma_y-\sigma+{}^1/_3T}{\tau_{yz}}+\varepsilon_z\\
&=\frac{\sigma}{3K}+\frac{1}{2G}\left[s_x+\sqrt{s_x^2-4\left(s_y s_z-\tau_{yz}^2\right)}\right]\\
&=\frac{\sigma}{3K}+\frac{1}{3G}\left[s_y+\sqrt{s_y^2-4\left(s_z s_x-\tau_{zx}^2\right)}\right]\\
&=\frac{\sigma}{3K}+\frac{1}{2G}\left[s_z+\sqrt{s_z^2-4\left(s_x s_y-\tau_{xy}^2\right)}\right]
\end{aligned}
\tag{2.13}
$$

Relations (2.13) may be obtained from Expressions (2.10) by the theory of the generalized elastic potential.

To the five equalities in (2.13) there must be added three equations of equilibrium

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0, \ \ldots \tag{2.14}$$

and, in addition, the relations (1.3).

The system of nine equations (2.13), (2.14) and (1.3) in the six stress components and three displacement components determines the behavior of the elastic model under consideration.

The original system of equations may be presented in another form. By substitution of Expressions (2.9) into Equation (2.14) we get

$$\begin{gathered}\frac{\partial\sigma}{\partial x} - T\sin\theta_1\left(\cos\theta_1\frac{\partial\theta_1}{\partial x} + \cos\theta_2\frac{\partial\theta_1}{\partial y} + \cos\theta_3\frac{\partial\theta_1}{\partial z}\right) - \\ - T\cos\theta_1\left(\sin\theta_1\frac{\partial\theta_1}{\partial x} + \sin\theta_2\frac{\partial\theta_2}{\partial y} + \sin\theta_3\frac{\partial\theta_3}{\partial z}\right) - \\ - \frac{1}{3}\frac{\partial T}{\partial x} + \cos\theta_1\left(\frac{\partial T}{\partial x}\cos\theta_1 + \frac{\partial T}{\partial y}\cos\theta_2 + \frac{\partial T}{\partial z}\cos\theta_3\right) = 0, \ \ldots\end{gathered} \tag{2.15}$$

To the three equations (2.15) one must add the condition

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1 \tag{2.16}$$

We rewrite Equations (2.13) in the form

$$\begin{gathered}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\cos\theta_2}{\cos\theta_1} + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\cos\theta_3}{\cos\theta_1} \\ = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\cos\theta_1}{\cos\theta_2} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\cos\theta_3}{\cos\theta_2} \\ = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\cos\theta_1}{\cos\theta_3} + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\cos\theta_2}{\cos\theta_3} + \frac{\partial w}{\partial z} = \frac{\sigma}{3K} + \frac{T}{3G}\end{gathered} \tag{2.17}$$

The system of equations (2.15), (2.17) and (1.3) relates the unknowns σ, T, θ_1, θ_2, θ_3, u, v and w.

In the case of correspondence of the state of stress with the hexagon sides (Fig. 1) we will have

$$\gamma_{\max} = \frac{4}{3G}\tau_{\max} \tag{2.18}$$

where $\tau_{\max}$ and $\gamma_{\max}$ are the respective maximum shear stress and strain. For this model of an isotropic elastic body the shear modulus is 3/4 G.

If two of the principal stresses are equal, the relation

$$\frac{1}{2}(\gamma_1 + \gamma_2) = \frac{1}{G}\tau_{\max}$$

holds, where γ_1 and γ_2 are the principal shear strains.

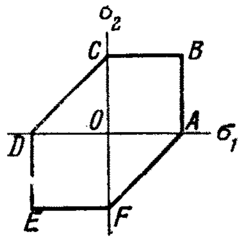

Fig. 2.

The Cartesian expressions corresponding to the hexagon sides are cumbersome and therefore have been omitted.

3. In the case of torsion $\sigma_1 = -\sigma_2$, $\sigma_3 = 0$, $\sigma = \epsilon = 0$; consequently the third invariant of the stress deviator tensor is zero.

For torsion, the stress-strain relations for an elastic isotropic body lead to generalized Hooke's Law for small strains for any potential function (1.1) if conditions (1.3) and (2.3) are accepted as valid.

Consider the case of plane strain. We suppose that

$$w = \varepsilon_z = \varepsilon_3 = \varepsilon_{xz} = \varepsilon_{yz} = \tau_{xz} = \tau_{yz} = 0$$

and that the remaining components depend only on the xy-coordinates.

The state of stress corresponds to the boundary AB (Fig. 1). We find from (2.2) that $\hat{\epsilon} = 0$; indeed, for plane strain the given elastic material behaves as though it were incompressible and independent of the modulus K.

It is easy to convince oneself that the remaining relations essentially coincide with those of a generalized Hooke's Law [1,2] for an incompressible body in plane strain

$$\varepsilon_x - \varepsilon_y = \frac{1}{3G}(\sigma_x - \sigma_y), \qquad \varepsilon_{xy} = \frac{2}{3G}\tau_{xy}, \qquad \varepsilon_x + \varepsilon_y = 0$$

4. We proceed to the case of plane stress. We assume that

$$\sigma_3 = \sigma_z = \tau_{xz} = \tau_{yz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{4.1}$$

All remaining components depend on the xy-coordinates. Figure 2 shows a section of a certain surface of equal levels of Φ in the plane $\sigma_3 = 0$. Evidently, we must differentiate between the cases where the state corresponds to the sides AB, BC, DE, EF, or to the sides CD, FA.

We consider first the side AB (Fig. 2). The sides CB, DE and EF are considered to be completely analogous.

Obviously, to derive the law of deformation one must start with the expression

$$U = \frac{\sigma^2}{2K} + \frac{1}{6G}(\sigma_1 - \sigma_2)^2 \tag{4.2}$$

We find from (4.2) that

$$\varepsilon_1 - \varepsilon = \frac{1}{3G}(\sigma_1 - \sigma_3), \qquad \varepsilon_2 - \varepsilon = 0, \qquad \varepsilon_3 - \varepsilon = -\frac{1}{3G}(\sigma_1 - \sigma_3) \tag{4.3}$$

Substitution of $\sigma_3 = 0$ into (4.3) gives

$$\varepsilon_1 = -\varepsilon = \frac{\sigma_1}{3G}, \qquad \varepsilon_2 - \varepsilon = 0, \qquad \varepsilon_3 - \varepsilon = -\frac{\sigma_1}{3G} \tag{4.4}$$

The principal stresses σ_1 and σ_2 $(\sigma_1 \geqslant \sigma_2)$ lie in the xy-plane and the principal strains ϵ_1 and ϵ_2 coincide with them in direction; therefore we have

$$\frac{\sigma_x - \sigma_y}{\varepsilon_x - \varepsilon_y} = \frac{\tau_{xy}}{\varepsilon_{xy}} \tag{4.5}$$

It is known that

$$\begin{aligned} \sigma_{1,2} &= \tfrac{1}{2}\left[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \\ \varepsilon_{1,2} &= \tfrac{1}{2}\left[(\varepsilon_x + \varepsilon_y) \pm \sqrt{(\varepsilon_x - \varepsilon_y)^2 + 4\varepsilon_{xy}^2}\right] \end{aligned} \tag{4.6}$$

We find, from (4.4) and (4.5), that

$$\begin{aligned} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + 4\varepsilon_{xy}^2} &= \frac{1}{6G}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \\ \varepsilon_x + \varepsilon_y - \sqrt{(\varepsilon_x - \varepsilon_y)^2 + 4\varepsilon_{xy}^2} &= \frac{2}{9K}(\sigma_x + \sigma_y) \end{aligned} \tag{4.7}$$

Supplementing the two equations of equilibrium and the condition (1.3) by the three relations (4.5) and (4.7), we obtain a system of six equations in the six unknowns σ_x, σ_y, τ_{xy}, u, v and w.

The system of equations may be linearized with the aid of the substitutions

$$\begin{aligned} \sigma_x &= \tfrac{3}{2}\sigma + \varkappa\cos 2\theta, \qquad & \varepsilon_x &= m + q\cos 2\theta \\ \sigma_y &= \tfrac{3}{2}\sigma - \varkappa\cos 2\theta, \qquad & \varepsilon_y &= m - q\cos 2\theta \\ \tau_{xy} &= \varkappa\sin 2\theta, \qquad & \varepsilon_{xy} &= q\sin 2\theta \end{aligned} \tag{4.8}$$

The condition (4.5) is satisfied identically and condition (4.7) takes the form

$$12Gq = 3\sigma + 2\varkappa, \qquad m - q = \frac{\sigma}{3K} \tag{4.9}$$

Substitution of Expressions (4.8) into the equations of equilibrium and compatibility gives a system of quasilinear equations in σ, κ, m, q and θ.

In the case where the state of stress corresponds to the cut DC or to AF, one must start with Expression (4.2) for the potential. It is easily shown that in this case the relations

$$\varepsilon_x - \varepsilon_y = \frac{1}{3G}(\sigma_x - \sigma_y), \qquad \varepsilon_{xy} = \frac{2}{3G}\tau_{xy}$$
$$\varepsilon_z = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y), \qquad \sigma_x + \sigma_y = \tfrac{9}{2}K(\varepsilon_x + \varepsilon_y) \tag{4.10}$$

hold.

The case of the axially symmetric problem is completely analogous. The relations may be obtained as a special case of the general problem.

5. We consider the case of a thin ring-shaped plate stretched by uniformly distributed forces applied around the edge (Fig. 3). Denote the inside radius by a, the outside radius by b, and the load intensity by p.

We set $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_\rho$. Since $\sigma_\theta > \sigma_\rho > 0$ everywhere in the plate, the state of stress corresponds to the side AB (Fig. 2).

We limit ourselves, for simplicity, to the case of incompressible material. We obtain from (4.4)

$$\varepsilon_\theta = \frac{\sigma_\theta}{3G}, \qquad \varepsilon_\rho = 0, \qquad \varepsilon_z = -\frac{\sigma_\theta}{3G} \tag{5.1}$$

It is known that

$$\varepsilon_\rho = \frac{du}{d\rho}, \qquad \varepsilon_\theta = \frac{u}{\rho}, \qquad \varepsilon_z = \frac{dw}{dz}$$

Here u and v are the displacements along the axes ρ and z.

The single equation of equilibrium is written in the form

$$\frac{d\sigma_\rho}{d\rho} + \frac{\sigma_\rho - \sigma_\theta}{\rho} = 0 \tag{5.2}$$

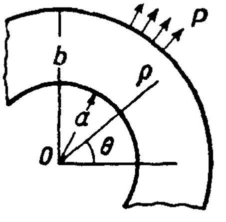

Fig. 3.

It is easy to obtain

$$\sigma_\rho = \frac{pb\ln(\rho/a)}{\rho\ln(b/a)}, \qquad \sigma_\theta = \frac{pb}{\rho\ln(b/a)}$$
$$\varepsilon_\rho = 0, \qquad \varepsilon_\theta = -\varepsilon_z = \frac{pb}{3G\rho\ln(b/a)} \tag{5.3}$$
$$u = \frac{pb}{3G\ln(b/a)}, \qquad w = \frac{pbz}{3G\rho\ln(b/a)}$$

We present for comparison the solution of the same problem obtained by using the linear relations of a generalized Hooke's Law [1]:

$$\sigma_\rho = \frac{pb^2}{b^2-a^2}\left(1-\frac{a^2}{\rho^2}\right), \qquad \sigma_\theta = \frac{pb^2}{b^2-a^2}\left(1+\frac{a^2}{\rho^2}\right)$$

$$\varepsilon_\rho = \frac{pb^2}{2G(b^2-a^2)}\left(\frac{1}{3}-\frac{a^2}{\rho^2}\right), \qquad \varepsilon_\theta = \frac{pb^2}{2G(b^2-a^2)}\left(\frac{1}{3}+\frac{a^2}{\rho^2}\right) \tag{5.4}$$

$$\varepsilon_z = -\frac{pb^2}{3G(b^2-a^2)}, \qquad u = \frac{pb^2}{2G(b^2-a^2)}\left(\frac{\rho}{3}+\frac{a^2}{\rho}\right)$$

$$w = -\frac{pb^2 z}{3G(b^2-a^2)}$$

We note that the stresses are rather close, the largest difference being in the displacements and strains.

6. Evidently, different consistent models of an elastic isotropic body may be assumed, which through experiments on uniaxial tension-compression and pure shear lead to linear relations between stresses and strains.

The model considered is one of the possible ones. The use of this or some other model is connected with the degree of correspondence between the theoretical results and the experimental data, and with what is no less important, the mathematical simplicity of the original equations.

It is possible that deviations of the behavior of an ideal elastic body from that of an elastic Hookean body in the region of small strains may be accounted for by passing to some other potential surface; nevertheless, the Hooke model leads unquestionably with the greatest simplicity and exactness to the well-studied linear equations of elliptic type.

We note also that in the light of Hencky's interpretation [8], the Mises plasticity condition as a certain energy of elastic shape change does not have exceptional value. For example, for certain conditions the energy of elastic shape change coincides with the plasticity condition of Tresca.

Analogously, for any given plasticity condition there may be assigned a model of an elastic isotropic body, for which the expression for energy of elastic shape change coincides under a certain interpretation with the given plasticity condition.

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